AD-A098 072    PITTSBURGH UNIV PA INST FOR STATISTICS AND APPLICATIONS    F/G 12/1
CHARACTERIZATIONS OF GENERALIZED MARKOV-POLYA AND GENERALIZED P--ETC(U)
FEB 81  K G JANARDAN, B R RAO    F49620-79-C-0161
UNCLASSIFIED  TR-81-03    AFOSR-TR-81-0386    NL

1 of 1
AD A
098072

END
DATE
FILMED
5-81
DTIC

LEVEL II

(2)

# CHARACTERIZATIONS OF GENERALIZED MARKOV-POLYA AND GENERALIZED POLYA-EGGENBERGER DISTRIBUTIONS

K. G. Janardan*
University of Pittsburgh
and
B. Raja Rao
University of Pittsburgh

February 1981
Technical Report No. 81-03

Institute for Statistics and Applications
Department of Mathematics and Statistics
University of Pittsburgh
Pittsburgh, PA 15260





*On Sabbatical leave from Sanganon State University, Springfield, IL 62708



Part of this work is sponsored by the Air Force Office of Scientific Research under Contract [redacted]. Reproduction in whole or in part is permitted for any purpose of the United States Government.

F49620-79-C-0161

# DISCLAIMER NOTICE

UNCLASSIFIED

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER: AFOSR-TR- 81-0386

2. GOVT ACCESSION NO.: AD-A098672

4. TITLE (and Subtitle)

Characterizations of Generalized Markov-Polya and Generalized Polya-Eggenberger Distributions

5. TYPE OF REPORT: INTERIM

6. PERFORMING ORG. REPORT NUMBER: U.P. Stat. Report 81-

7. AUTHOR(s)

K. G. Janardan & B. Raja Rao

8. CONTRACT OR GRANT NUMBER: F49620-79-C-0161

9. PERFORMING ORGANIZATION NAME AND ADDRESS

University of Pittsburgh
Dept. of Math. & Statistics
Pittsburgh, PA 15260

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2304/A5 61102F

11. CONTROLLING OFFICE NAME AND ADDRESS

AFOSR/NM
Bolling AFB, DC 20332

12. REPORT DATE: February 1981

13. NUMBER OF PAGES: 13

14. MONITORING AGENCY NAME & ADDRESS

15. SECURITY CLASS: UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT

18. SUPPLEMENTARY NOTES

19. KEY WORDS

Generalized Markov-Polya distribution; ... tion; ... conditional distribution characterizations.

20. ABSTRACT

A ... to partial destruction ... A characterization of the generalized ... in the context of the Rao-Rubin condition. ... theorems are also proved concerning these probability distributions.

DD FORM 1473

UNCLASSIFIED

CHARACTERIZATIONS OF GENERALIZED MARKOV-POLYA AND GENERALIZED POLYA-EGGENBERGER DISTRIBUTIONS

K. G. Janardan* and B. Raja Rao
Department of Mathematics & Statistics
University of Pittsburgh
Pittsburgh, PA 15260

ABSTRACT

A discrete model is considered where the original observation is subjected to partial destruction according to the generalized Markov-Polya damage model. A characterization of the generalized Polya-Eggenberger distribution is given in the context of the Rao-Rubin condition. Several other characterization theorems are also proved concerning these probability distributions

Key Words & Phrases: Generalized Markov-Polya distribution; generalized Polya-Eggenberger distribution; damaged model; conditional distribution characterizations.



* On Sabbatical leave from Sangamon State University Springfield, IL 62703

## 1. INTRODUCTION

Urn models are readily adapted to the development of probability distributions used in the analysis of complex problems in real life situations. In many natural phenomena involving individuals or living organisms, the probability of success seems to increase or decrease with the number of successes or failures. Thus, with the aid of urn models, research workers have developed a number of discrete probability distributions (see Johnson and Kotz, 1977; chp. 4) when the probability of success of an event is a linear function of the number of successes. Among the principal researchers who have used urn models for developing discrete probability models for contagious events, Markov (1917) and Polya and Eggenberger (1923) are pioneers in the field.

Following Wald's (1960) approach, Janardan (1973), and Janardan and Schaeffer (1977) have recently considered a new three urn model with predetermined strategy and have derived the generalized Markov-Polya distribution (GMPD) as a model (1.1) for voting in small groups where contagion is present within each group and the group leader devises some new strategies for bringing success to his candidate:

$$P(X=x) = \binom{n}{x} \frac{a\, b(a+b+nt)\, (a+xt)^{(x,c)} (b+(n-x)t)^{(n-x,c)}}{(a+b)(a+xt)(b+(n-x)t)(a+b+nt)^{(n,c)}} \qquad (1.1)$$

where $a>0$, $b>0$, $0 \le t \le 1$, $c \neq 0$, and $x = 0, 1, 2, \ldots, n$. If $t = 0$ this reduces to Markov-Polya distribution (see equation (5.1) of Johnson-Kotz, 1977, p. 177). [illegible] the Polya-Eggenberger distribution.

The GMPD (1.1) has several interesting properties and a large number of applications (see Janardan and Schaeffer, 1977 and Janardan, 1978). Under certain limiting conditions, the probability distribution (1.1) gives the limiting form:

$$P(X=x) = \frac{b(b+xt)^{(x,c)}}{(b+xt)\, x!\, c^x}\, \theta^x (1-\theta)^{(b+xt)/c} \qquad (1.2)$$

where $x=0,1,2,\ldots$, $h>0$, $0<\beta<1$, $0\leq t\leq 1$ and $c\neq 0$.

This distribution was named "The generalized Polya-Eggenberger" distribution (GPED) by Janardan (1973) since (1.2) reduces to the Polya-Eggenberger distribution when $t=0$ (see Patil and Joshi, 1968 p. 20).

Situations often arise where the original observation produced by nature undergoes a destructive process and what is recorded is only the damaged portion of the actual (original) observation. This problem was first brought to light by Rao (1963) when he considered the resultant models after the observations, produced by some probability model, were ruined by other probability models. Subsequently, Rao and Rubin (1964) proved that if the observation generated by nature (denoted by r.v. X) is reduced to Y by a binomial destructive process and if Y satisfies the condition:

$$P(Y=k) = P(Y=k/\text{no damage}) = P(Y=k/\text{partial damage}), \tag{1.3}$$

then the original r.v. X must have the Poisson distribution. In the literature, this result is known as Rao-Rubin characterization of the Poisson distribution and the condition (1.3) is called RR-condition.

In this paper, we consider the generalized Markov-Polya distribution as a damage model subject to RR condition and characterize the generalized Polya-Eggenberger distribution (GPED) as a model of contagion for the production of observations in nature. In addition to this result which is given in [illegible], we prove several other theorems which characterize the generalized Markov-Polya and generalized Polya-Eggenberger distributions.

## 2. NOTATION AND IDENTITIES

To begin with, we shall discuss the notation and two identities required in this paper. The notation $a^{(x,c)}$ used in the definitions of probability distributions (1.1) and (1.2), and in sequel stands for the ascending factorial

$$
\begin{aligned}
m^{(x,c)} &= m(m+c)(m+2c)\dots(m+(x-1)c), \\
m^{(x,0)} &= m^x, \quad m^{(0,c)}=1, \\
m^{(x,-1)} &= m^{(x)}=m(m-1)\dots(m-x+1), \\
m^{(x,+1)} &= m^{[x]}=m(m+1)\dots(m+x-1).
\end{aligned}
\tag{2.1}
$$

From Janardan (1973), we record the following two identities:

$$\sum_{k=0}^{n} J_k(A,t,c)J_{n-k}(B,t,c) = J_n(A+B,t,c) \tag{2.2}$$

where $J_m(a,t,c) = a(a+mt)^{(m,c)}/(a+mt)m!$ (2.3)

and $J_o(a,b,c) = 1.$

$$\sum_{k=0}^{\infty} J_k(A,t,c)V^k = W^{-A/c}, \quad \text{with } V=(1-W)W^t/c. \tag{2.4}$$

These identities can be derived by using Lagrange's expansion. With this notation, the probability distributions (1.1) and (1.2) can be written respectively as

$$P(X=x) = J_x(a,t,c)J_{n-x}(b,t,c)/J_n(a+b,t,c), \tag{2.5}$$

$$P(X=x)=f_x = J_x(b,t,c)(\beta/c)^x(1-\beta)^{(b+xt)/c} \tag{2.6}$$

## 3. CHARACTERIZATIONS BASED ON CONDITIONAL DISTRIBUTIONS

We now prove the following theorem:

**THEOREM 1:** If X and Y are two independent random variables having the generalized Polya-Eggenberger distributions with parameters $(a,t,c,\beta)$ and $(b,t,c,\beta)$ respectively, then the conditional distribution of X given X+Y=n is a generalized Markov-Polya distribution as given in (1.1).

**PROOF:** By definition of conditional probability,

$$P(X=x/X+Y=n) = P(X=x,Y=n-x)/P(X+Y=n)$$

$$= \frac{J_x(a,t,c)(\beta/c)^x(1-\beta)^{(a+xt)/c}][J_{n-x}(b,t,c)(\beta/c)^{n-x}(1-\beta)^{(b+(n-x)t)/c}]}{\sum_{x=0}^{n}[J_x(a,t,c)(\beta/c)^x(1-\beta)^{(a+xt)/c}][J_{n-x}(b,t,c)(\beta/c)^{n-x}(1-\beta)^{(b+(n-x)t)/c}]}$$

$$= \frac{J_x(a,t,c)J_{n-x}(b,t,c)}{\sum_{x=0}^{n} J_x(a,t,c)J_{n-x}(b,t,c)}$$

By identity (2.2), the denominator equals $J_n(a+b,t,c)$, which proves the theorem. The following theorem gives the converse of the above.

THEOREM 2: Let X and Y be two independent discrete r.v's such that the conditional distribution of X=x given X+Y=n is the generalized Markov-Polya distribution given by (1.1) or (2.5) then each X and Y has a generalized Polya-Eggenberger distribution as in (1.2).

PROOF: By hypothesis of the theorem, $P(X=x/X+Y=n)$ is given by $J_x(a,t,c)J_{n-x}(b,t,c)\ J_n(a,b,t,c)$ which is of the form $\alpha(x)\beta(n-x)/\gamma(n)$ with $\alpha(x) = J_x(a,t,c)$, $\beta(n-x)=$ $J_{n-x}(b,t,c)$ and $\gamma(n) = J_n(a+b,t,c)$.

Applying theorem 1 of Janardan (1975), we get

$f(x) = p\alpha(x)e^{rx}$ and $g(y) = q\beta(y)e^{ry}$, where p, q and r are some positive constants. Setting $e^r = \beta(1-\beta)^{t/c}$, the functions $f(\cdot)$ and $g(\cdot)$ can be written as

$$f(x) = pJ_x(a,t,c)\ \beta^x(1-\beta)^{tx/c}, \text{ and}$$

$$g(y) = qJ_y(b,t,c)\ \beta^y(1-\beta)^{ty/c}.$$

Since $1 = \sum_{x=0}^{\infty} f(x) = \sum_{y=0}^{\infty} g(y)$, applying the identity (2.4), we get

$p= (1-\beta)^{-a/c}$ and $q = (1-\beta)^{-b/c}$ completing the proof of the theorem.

THEOREM 3 : If a non-negative integer random variable $N$ is sub-divided into two components $N_A$ and $N_B$ such that the conditional distribution $P(N_A = x, N_B = n-x/N=n)$ is the generalized Markov-Polya distribution (2.5) then the r.v's $N_A$ and $N_B$ are independent if, and only if, $N$ has a generalized Polya-Eggenberger distribution.

PROOF: The joint probability of $N_A$ and $N_B$ becomes

$$P[N_A=x, N_B=n-x] = \frac{J_x(a,t,c)J_{n-x}(b,t,c)}{J_n(a+b,t,c)} \cdot P(N=n) . \qquad (3.1)$$

If $N$ has a generalized Polya-Eggenberger distribution, then with $h = a+b$, its probability function is

$$P(N=n) = J_n(a+b,t,c)(\beta/c)^n(1-\beta)^{(a+b+nt)/c} , \qquad (3.2)$$

$$a>0,\ b>0,\ 0\leq t\leq 1,\ 0<\beta<1,\ c\neq 0.$$

Inserting the value of $P(N=n)$, we can easily write (3.1) as

$$P[N_A=x, N_B=n-x] =$$

$$J_x(a,t,c)(\beta/c)^x(1-\beta)^{(a+xt)/c}J_{n-x}(b,t,c)(\beta/c)^{n-x}(1-\beta)^{(b+(n-x)t)/c}$$

$$= P(N_A = x)\ P(N_B = n-x).$$

Conversely, if $N_A$ and $N_B$ are independent r.v's such that the conditional distribution of $N_A$ and $N_B$ given $N_A + N_B = n$ is the GMPD, then $N_A$ and $N_B$ have the generalized Polya-Eggenberger distributions. This follows from theorem 2 .

THEOREM 4: If X and Y are two independent r.v's defined on non-negative integers such that $P(X=x) = f(x)>0, \sum_{x=0}^{\infty} f(x) = 1$ and

$P(Y=y) = g(y)>0,$

$\sum_{y=0}^{\infty} g(y) = 1$ and further if for $a_n/b_n = A$,

$P(X=k/X+Y=n) =$

$$\binom{n}{k} \frac{a_n b_n (A+nt)(a_n+kt)^{(k,c)}(b_n+(n-k)t)^{(n-k,c)}}{(a_n+kt)(b_n+(n-k)t)\,A(A+nt)^{(n,c)}} \tag{3.3}$$

for $k = 0,1,2,\ldots,n$.

$= 0$ for $k>n$

Then (i) $a_n$ is independent of $n$ and equals a constant '$a$' for all values of $n$, and

(ii) $X$ and $Y$ must have generalized Polya-Eggenberger distributions with parameters $(a,t,c,\beta)$ and $(b,t,c,\beta)$ respectively.

<u>PROOF</u>: Since $X$ and $Y$ are independent r.v's we have

$$P(X=k/X+Y=n) = f(k)g(n-k) \Big/ \sum_{k=0}^{n} f(k)g(n-k) \,. \tag{3.4}$$

Using (3.3) and (3.4) we can derive the functional relation (3.5) for all values of $0 \le k \le n$, and $n \ge 1$:

$$\frac{f(k)g(n-k)}{f(k-1)g(n-k+1)} = \tag{3.5}$$

$$\frac{n-k+1}{k} \; \frac{(a_n+kt)^{(k,c)}(b_n+(n-k)t)^{(n-k,c)}(a_n+(k-1)t)(b_n+(n-k+1)t)}{(a_n+(k-1)t)^{(k-1,c)}(b_n+(n-k+1)t)^{(n-k+1,c)}(a_n+kt)(b_n+(n-k)t)}$$

Replacing $k$ by $k+1$ and $n$ by $n+1$ in (3.5) and dividing the resulting expression by (3.5) we get $f(k+1)f(k-1)/f^2(k)$ on the left side and a very complex untidy expression on the right side. Since the left side is independent of $n$, the right side must also be independent of $n$. Thus, $a_{n+1}=a_n=a$ and $b_{n+1}=b_n=b$ for all $n$, and hence ignoring the subscripts on $a$'s and $b$'s we will have

$$\frac{f(k+1)f(k-1)}{f^2(k)} =$$

$$\frac{k}{k+1} \; \frac{(a+(k-1)t)^{(k-1,c)}(a+(k+1)t)^{(k+1,c)}(a+kt)^2}{(a+(k-1)t)(a+(k+1)t)[(a+kt)^{(k,c)}]^2} \,, \tag{3.6}$$

which by continued substitution for $k=1,2,\ldots,(n-1)$, and multip-

lication together yields

$$\frac{f(n)f(0)}{f(n-1)f(1)} = \frac{1}{n!}\frac{(a+nt)^{(n,c)}(a+(n-1)t)}{a(a+nt)(a+(n-1)t)^{(n-1,c)}} . \tag{3.7}$$

Setting $B = f(1)/f(0)a$, the recurrence relation (3.7) becomes

$$f(n) = \frac{B\ (a+nt)^{(n,c)}(a+(n-1)t)}{n!\ (a+nt)(a+(n-1)t)^{(n-1,c)}}\ f(n-1), \tag{3.8}$$

which is true for all integral $n$. Thus,

$$f(n) = B^n a(a+nt)^{(n,c)} f(0)/(a+nt)\ n! \tag{3.9}$$

Since $\sum_{n=0}^{\infty} f(n) = 1$, the series (3.9) must converge to unity. Let the unknown positive quantity $B$ be equal to $(\beta/c)(1-\beta)^{t/c}$, $0<\beta<1, t>0$, and $c \neq 0$. Thus,

$$1 = \sum_{n=0}^{\infty} \frac{a(a+nt)^{(n,c)}}{(a+nt)n!}\left(\frac{\beta}{c}\right)^n (1-\beta)^{nt/c} f(0) .$$

By applying identity (2.4), we can easily see that $f(0) = (1-\beta)^{-a/c}$ and $f(x) = J_x(a,t,c)(\beta/c)^x(1-\beta)^{(a+tx)/c}$ .

which proves that the r.v.X is distributed as the GPED with parameters $(a,t,c,\beta)$. By putting $k=1$ in (3.5), one can easily see that

$$\frac{g(n)}{g(n-1)} = \frac{B\ (b+nt)^{(n,c)}(b+(n-1)t)}{n!\ (b+nt)(b+(n-1)t)^{(n-1,c)}} .$$

Hence $g(n) = B^n b(b+nt)^{(n,c)} g(0)/(b+nt)n!$ and the fact $\sum_{n=0}^{\infty} g(n) = 1$ will similarly give $g(0)=(1-\beta)^{-b/c}$ for $B = (\beta/c)(1-\beta)^{t/c}$. Thus, the r.v.Y must be the GPED with parameters $(b,t,c,\beta)$.

REMARK: It was shown in theorem 3 , that if X and Y are independent generalized Polya-Eggenberger random variables, then the conditional distribution of X given X+Y is generalized Markov-

Polya distribution. The above theorem, which was motivated by theorem 1 of Chatterji (1963), shows that a weak form of this property characterizes the [illegible].

## 4. CHARACTERIZATION THEOREMS BASED ON THE RR CONDITION

Let (X,Y) be a random vector of non-negative integer-valued components such that

$$P(X=n, Y=k) = f_n S(k/n) \tag{4.1}$$

where $\{f_n : n=0,1,2,\ldots\}$ and $\{S(k/n): k=0,1,2,\ldots,n\}$ for each $n \geq 0$ are discrete probability distributions. That is, the marginal distribution of X is $\{f_n\}$ and for each $n \geq 0$ with $f_n > 0$, the conditional distribution of Y given X=n is $\{S(k/n): k=0,1,2,\ldots,n\}$. Further,

$$P\{Y=k/\text{no damage}\} = f_k S(k/k) / \sum_{j=0}^{\infty} f_j S(j/j) \tag{4.2}$$

$$P\{Y=k/\text{damaged}\} = \sum_{n=k+1}^{\infty} f_n S(k/n) / \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} S(k/n). \tag{4.3}$$

<u>THEOREM 5</u>: If a r.v. X defined on non-negative integers is distributed in nature as a GPFD (1.2) with parameters $(a+b, t, c, \beta)$ and if it is damaged and reduced to k by the GMD (1.1) and further, if Y is the resulting random variable, then (i) RR Condition (1.3) is satisfied, and (ii) has a GPFD with parameters $(a, t, c, \beta)$.

<u>PROOF</u>: $P(Y=k) = \sum_{n=k}^{\infty} f_n S(k/n)$ because P(X=n) = [illegible]

$$\sum_{n=k}^{\infty} [J_n(a+b,t,c)(\beta/c)^n (1-\beta)^{(a+b+nt)/c}\, J_k(a,t,c) J_{n-k}(b,t,c) / J_n(a+b,t,c)]$$

$$= J_k(a,t,c)(\beta/c)^k (1-\beta)^{(a+kt)/c} [\sum_{n=k}^{\infty} J_{n-k}(b,t,c)(\beta/c)^{n-k} (1-\beta)^{(b+(n-k)t)/c}]$$

$= J_k(a,t,c)(\beta/c)^k (1-\beta)^{(a+kt)/c}$ since the sum of the terms in the square brackets is one.

From equation (4.2),

$$P(Y=k/\text{no damage}) = J_k(a,t,c)(f/c)^k(1-f)^{(a+kt)/c} = P(Y=k).$$

From equation (4.3),

$$P(Y=k/\text{damaged}) = \frac{J_k(a,t,c)(f/c)^k(1-f)^{(a+kt)/c}}{\sum_{k=0}^{\infty} J_k(a,t,c)(f/c)^k(1-f)^{(a+kt)/c}}$$

$$= P(Y=k)\ .$$

<u>THEOREM 6</u>: Let X be a non-negative integer-valued r.v. and let the probability that an observation n of X is reduced to k during a destructive process be given by the GMPD:

$$S(k/n) = J_k(a,t,c)J_{n-k}(b,t,c)/J_n(1,t,c),$$

for $0<a<1$, $a+b=1$, $0\le t<1$, $c>0$, and $k=0,1,2,\ldots n$. If the resulting r.v. Y is such that it satisfies the RR-Condition (1.3), then X has a GPED with parameters $(1,t,c,f)$.

<u>PROOF</u>: The RR-Condition is equivalent to

$$\sum_{n=k}^{\infty} f_n S(k/n) = f_k S(k/k) / \sum_{j=0}^{\infty} f_j S(j/j),$$

where $f_k = P(X=k)$. This gives

$$\sum_{n=k}^{\infty} f_n \frac{J_k(a,t,c)J_{n-k}(b,t,c)}{J_n(1,t,c)} = \frac{f_k J_k(a,t,c)/J_k(1,t,c)}{\sum_{j=0}^{\infty} f_j J_j(a,t,c)/J_j(1,t,c)}\ . \qquad (4.4)$$

setting $n=k+s$ and concelling $J_k(a,t,c)$ on both sides we get

$$\sum_{s=0}^{\infty} f_{k+s}\frac{J_s(b,t,c)}{J_{k+s}(1,t,c)} = \frac{f_k/J_k(1,t,c)}{\sum_{j=0}^{\infty} f_j J_j(a,t,c)/J_j(1,t,c)} \qquad (4.5)$$

$$\text{Define } f_k = F(k)J_k(1,t,c)V^k, \tag{4.6}$$

for all integral values of k, where V is some arbitrary quantity to be set later. Substituting (4.6) in (4.5) we get

$$\sum_{s=0}^{\infty} F(k+s)J_s(b,t,c)V^{k+s} = F(k)V^k / \sum_{j=0}^{\infty} F(j)J_j(a,t,c)V^j \tag{4.7}$$

Let $G(az,t) = \sum_{k=0}^{\infty} F(k)J_k(az,t,c)V^k$ ,

where $-\infty < z < \infty$ so that $G(0,t) = F(0)$ and $G(1,t) = 1$. Multiplying both sides of (4.7) by $J_k(az,t,c)$ and summing over k from 0 to $\infty$ (4.8) becomes

$$\sum_{n=0}^{\infty} F(n)V^n \{ \sum_{k=0}^{n} J_k(az,t,c)J_{n-k}(b,t,c) \} = G(az,t)/G(a,t) \ . \tag{4.9}$$

By identity (2.2) the inner sum on the left side of (4.9) is equivalent to $J_n(az+b,t,c)$, and hence (4.9) gives the bivariate functional equation:

$$G(az+b,t)G(a,t) = G(az,t). \tag{4.10}$$

Clearly $G(a+b,t) = G(1,t) = 1$. Setting $x=a(z-1)$ and $b=1-a$ (4.10) gives

$$G(x+1,t)G(a,t) = G(x+a,t) \tag{4.11}$$

Setting $\phi(x) = G(x+1,t)$, and $a-1=y$ (4.11) reduces to the Cauchy functional equation, $\phi(x)\phi(y) = \phi(x+y)$, whose non-trivial solution is given by $\phi(x) = e^{\lambda x}$. Thus, $G(x,t) = e^{\lambda(x-1)}$ which is the probability generating function of the Poisson distribution. Now replacing x by az, assigning a value $(1-e^{-\lambda c})e^{-\lambda t}/c$ to V, using the definition of G(az,t), we get

$$e^{\lambda az} = \sum_{k=0}^{\infty} F(k)e^{\lambda} J_k(az,t,c) \ V^k \tag{4.12}$$

To determine the value of F(k), consider the identity (2.4) with $A = az$ and $w = e^{-\lambda c}$ which gives

$$e^{\lambda az} = \sum_{k=0}^{\infty} J_k(az,t,c)\, V^k . \qquad (4.13)$$

Subtracting (4.13) from (4.12),

$$0 = \sum_{k=0}^{\infty} [F(k)e^{\lambda}-1]J_k(az,t,c)\, V^k . \qquad (4.14)$$

Since (4.14) is true for all values of $\lambda$, it is obvious that $F(k)=e^{-\lambda}$ for all k. Hence, by definition (4.6), we get

$$\begin{aligned} f(k) &= F(k)\, J_k(1,t,c)\, V^k \\ &= e^{-\lambda} J_k(1,t,c)[e^{-\lambda t}(1-e^{-\lambda c})/c]^k \\ &= J_k(1,t,c)(\beta/c)^k(1-\beta)^{(1+kt)/c} \end{aligned}$$

$$\text{with } \beta=1-e^{-\lambda c} .$$

That is, X has a GPED with parameter $(1,t,c,\beta)$.


## ACKNOWLEDGEMENTS

The first author thanks professor P.R.Krishniah for all the facilities provided to him while the author was spending his sabbatical leave at the University of Pittsburgh. Miss Beth Ford typed the manuscript.

## BIBLIOGRAPHY


(1). Chatterji, S. D. (1963). Some elementary characterizations of the Poisson distribution. Amer. Math. Monthly 70, 958-64.

(2). Hald, A. (1960). The Compound hypergeometric distribution and a system of single sampling inspection plans based on prior distributions and costs. Technometrics 2, 275-340.

(3). Janardan, K. G. (1973). A four urn model with predetermined strategy. Technical Report, Sangamon State Universiry, Springfield, IL

(4). Janardan, K. G. (1975). Characterizations of certain discrete distributions. Statistical Distributions in Scientific Work 3, (Eds.Patil et.al) D. Reidel Publishing Co. Holland. pp. 359-64.

(5). Janardan, K. G. (1978). On a generalized Markov-Polya distribution. Gujarat Statistical Review 5 (1), 16-32.

(6). Janardan, K. G. & Schaeffer, D. J. (1977). A generalization of Markov-Polya distribution: Its extensions and applications. Biom. J 19 (2), 87-106.

(7). Johnson, N. L. & Kotz, S. (1977). Urn models and their applications. John Wiley & Sons Inc., New York and Canada.

(8). Markov (1917). On limiting formulas for computing probabilities (O. Nekotovih Predelnih Formulah Iscislenia Veroiatnostei-Russian). Izvestia Imperatorsky Academy Nauk, Seria VI, II (3), 177-86.

(9). Patil, G. P. & Joshi, S. W. (1968). A Dictionary and Bibliography of Discrete Distributions, Oliver & Boyd, Edinburgh.

(10). Polya, G. & Eggenberger, F. (1923). Über die Statistik verketteter vorgange. Zeitschrift fur Angewandte Mathematik and Mechanik 3, 279-89.

(11). Rao, C. R. (1963). On discrete distributions arising out of methods of ascertainment. Int. Symp. Discrete Distributions, Montreal.

(12). Rao, C. R. & Rubin, H. (1964). On a characterization of the Poisson distribution. Sankhya A 26, 295-98.

ATE

LMED

-8